HYPERELLIPTICITY AND KLEIN BOTTLE COMPANIONSHIP IN SYSTOLIC GEOMETRY

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Abstract. Given a hyperelliptic Klein surface, we construct companion Klein bottles. Bavard's short loops on companion bottles are studied in relation to the surface to improve an inequality of Gromov's in systolic geometry.

Contents

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1. Introduction

Systolic inequalities for surfaces compare length and area, and can therefore be thought of as "opposite" isoperimetric inequalities. The study of such inequalities was initiated by C. Loewner in '49 when he proved his torus inequality for $\mathbb{T}^2$. In higher dimensions, we have M. Gromov's deep result [12] on the existence of a universal upper bound for the systole in terms of the volume of an essential manifold. A promising research direction, initiated by L. Guth [16, 17] is a search for a proof of Gromov's bound without resorting to filling invariants as in [12].

In dimension 2, the focus has been, on the one hand, on obtaining near-optimal asymptotic results in terms of the genus [21, 23], and on the other, on obtaining realistic estimates in cases of low genus [22, 18]. One goal has been to determine whether all aspherical surfaces satisfy Loewner's bound, a question that is still open in general. It was resolved in the affirmative for genus 2 in [22]. An older optimal inequality of C. Bavard [3] for the Klein bottle K is stronger than Loewner's bound. Other than these results, no bound is available that's stronger than Gromov's general estimate [12] for aspherical surfaces:

$$\mathrm{sys}^2 \leq \frac{4}{3}\,\mathrm{area}, \tag{1.1}$$

even for the genus 3 surface. There are at most 17 genera where Loewner's bound could be violated [21], including genus 3.

As Gromov points out in [12], the $\frac{4}{3}$ bound (1.1) is actually *optimal* in the class of Finsler metrics. Therefore any further improvement is not likely to result from a simple application of the coarea formula. One can legitimately ask whether any improvement is in fact possible, of course in the framework of Riemannian metrics.

Our purpose in the present article is to furnish such an improvement in the case of the surface

$$K\#\mathbb{RP}^2 = \mathbb{T}^2\#\mathbb{RP}^2 = 3\mathbb{RP}^2$$

of Euler characteristic -1, as follows.

Theorem 1.1. *The surface* $3\mathbb{RP}^2$ *satisfies the bound*

$$\mathrm{sys}^2 \leq 1.333\ \mathrm{area}$$

(see Theorem 15.1 below).

Note the absence of an ellisis following "333", making our estimate an improvement on Gromov's $\frac{4}{3}$ bound (1.1). Our proof exploits a variety of techniques ranging from hyperellipticity to the coarea formula and

cutting and pasting. An inspection of the proof reveals that all the estimates are very tight and only produce an improvement in the fourth decimal place, a fact the present writer has no explanation for other than a remarkable coincidence.

Note that the current best upper bound for the systole only differs by about 30% from the Silhol-Parlier hyperblolic example (see Section 15).

2. Hyperellipticity

Complex-analytic information can be applied to the study of Klein surfaces and their Riemannian geometry. The argument involves dragging a short loop across, from the extreme right to the extreme left of the diagram

$$3\mathbb{RP}^2 \leftarrow \Sigma_2 \to S^2 \leftarrow \mathbb{T}_{a,b} \to K.$$

(see (6.3) for details). The quadratic equation

$$y^2 = p$$

over $\mathbb{C}$ is well known to possess two distinct solutions for every $p \neq 0$, and a unique solution for $p = 0$. Similarly, the locus (solution set) of the equation

$$y^2 = p(x) \tag{2.1}$$

for $(x, y) \in \mathbb{C}^2$, where $p(x)$ is a (generic) polynomial of even degree

$$2g + 2,$$

defines a Riemann surface which is a branched two-sheeted cover of $\mathbb{C}$. Such a cover is constructed by projection to the x-coordinate. The branching locus corresponds to the roots of $p(x)$. It is known that there exists a unique smooth closed Riemann surface Σ_g naturally associated with (2.1), sometimes called the *smooth completion* of the affine surface (2.1), together with holomorphic map

$$P_x : \Sigma_g \to \hat{\mathbb{C}} = S^2 \tag{2.2}$$

extending the projection to the x-coordinate. By the Riemann-Hurwitz formula, the genus of the smooth completion is g. All such surfaces are hyperelliptic by construction, where the hyperelliptic involution

$$J : \Sigma_g \to \Sigma_g$$

flips the two sheets of the double cover of S^2.

3. A pair of involutions

A hyperelliptic closed Riemann surface Σ_g admitting an orientation-reversing (antiholomorphic) involution τ can always be reduced to the form (2.1) where $p(x)$ is a polynomial all of whose coefficients are real, where the involution

$$\tau : \Sigma_g \to \Sigma_g$$

restricts to complex conjugation on the affine part of the surface in $\mathbb{C}^2$, namely $\tau(x, y) = (\bar{x}, \bar{y})$.

The special case of a fixed point-free involution τ can be represented as the locus of the equation

$$-y^2 = \prod_i (x - x_i)(x - \bar{x}_i), \tag{3.1}$$

where $x_i \in \mathbb{C} \setminus \mathbb{R}$ for all i. Here the minus sign on the left hand side ensures the absence of real solutions, and therefore the fixed point-freedom of τ.

By the uniqueness of the hyperelliptic involution, we have the commutation relation

$$\tau \circ J = J \circ \tau.$$

4. Klein surfaces and bottles

A Klein surface is a non-orientable closed surface. Such a surface can be thought of as an antipodal quotient Σ_g/τ of an orientable surface by a fixed-point free, orientation-reversing involution τ. It is known that (Σ_g, τ) can be thought of as a real surface. A Klein surface is homeomorphic to the connected sum

$$\mathbb{RP}^2 \# \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2 = n\mathbb{RP}^2$$

of n copies of the real projective plane. The case $n = 2$ corresponds to the Klein bottle $K = 2\mathbb{RP}^2$. The orientable double cover of the Klein bottle is a torus $\mathbb{T}^2$. The Klein bottle K can be thought of as a pair $(\mathbb{T}^2, \tau)$, or more precisely the quotient

$$\mathbb{T}^2/\{1, \tau\}$$

where τ is a fixed point free, orientation-reversing involution. We will sometimes use the abbreviated notation

$$K = 2\mathbb{RP}^2 = \mathbb{T}^2/\tau,$$

and refer to it as the antipodal quotient. In the case $n = 3$, we obtain the surface $3\mathbb{RP}^2$ of Euler characteristic -1, whose orientable double

cover is the genus 2 surface Σ_2. Thus, the surface $3\mathbb{RP}^2$ can be thought of as the pair (Σ_2, τ), or more precisely the quotient

$$3\mathbb{RP}^2 = \Sigma_2/\{1, \tau\},$$

where τ is a fixed point free, orientation-reversing involution. We will sometimes use the abbreviated notation

$$3\mathbb{RP}^2 = \Sigma_2/\tau,$$

and refer to it as the antipodal quotient.

5. ?-sided loops

A loop on a surface is called 2-sided if its tubular neighborhood is homeomorphic to an annulus, and 1-sided if it is homeomorphic to a Mobius strip. A one-sided loop

$$\gamma \subset 3\mathbb{RP}^2$$

lifts to a path on (Σ_2, τ) connecting a pair of points which form an orbit of the involution τ. In other words, the inverse image of γ under the double cover $\Sigma_2 \to 3\mathbb{RP}^2$ is a circle (i.e. has a single connected component homeomorphic to a circle). Meanwhile, a two-sided loop

$$\delta \subset 3\mathbb{RP}^2$$

lifts to a closed curve on (Σ_2, τ). In other words, the inverse image of δ under the double cover $\Sigma_2 \to 3\mathbb{RP}^2$ has a pair of connected components (circles).

6. Companionship

Given a real Riemann surface (Σ_g, τ), consider the presentation (3.1) with p real. We can write the roots of p as a collection of pairs $(a, \bar{a})$. In the genus 2 case, we have three pairs $(a, \bar{a}, b, \bar{b}, c, \bar{c})$. Thus the affine form of the surface is the locus of the equation

$$-y^2 = (x-a)(x-\bar{a})(x-b)(x-\bar{b})(x-c)(x-\bar{c}) \tag{6.1}$$

in $\mathbb{C}^2$. Choosing two conjugate pairs, for instance $(a, \bar{a}, b, \bar{b})$, we can construct a companion surface

$$-y^2 = (x-a)(x-\bar{a})(x-b)(x-\bar{b}), \tag{6.2}$$

By the Riemann-Hurwitz formula, we have $g = 1$ and therefore the (smooth completion of the) companion surface is a torus. We will denote it

$$\mathbb{T}_{a,b}.$$

By construction, its set of zeros is τ-invariant. In other words, the (affine part in $\mathbb{C}^2$ of the) torus is invariant under the action of complex conjugation. Thus, the surface $\mathbb{T}_{a,b}/\tau$ is a Klein bottle K, which we will refer to as a companion Klein bottle of the original Klein surface $3\mathbb{RP}^2 = \Sigma_2/\tau$, namely the antipodal quotient of (6.1).

The maps constructed so far can be represented by the following diagram of homomorphisms (note that two out of four point lefttward):

$$3\mathbb{RP}^2 \leftarrow \Sigma_2 \rightarrow S^2 \leftarrow \mathbb{T}_{a,b} \rightarrow K. \tag{6.3}$$

7. Equator

Complex conjugation on $\hat{\mathbb{C}} = S^2$ fixes a circle called the equator, which could be denoted

$$\hat{\mathbb{R}} \subset \hat{\mathbb{C}}.$$

The inverse image of the equator $\hat{\mathbb{R}}$ under the double cover $\Sigma_2 \to S^2$ is also a circle. We will refer to it as the equatorial circle, or equator, of Σ_2.

Lemma 7.1. *The equator of Σ_2 is the fixed point set of the composed involution $\tau \circ J = J \circ \tau$. The equator is invariant under the action of τ.*

The action of τ on the equator of Σ_2 is fixed point-free and thus can be thought of as a rotation by π.

In the case of the double cover $\mathbb{T}_{a,b} \to S^2$, the inverse image of the equator is a pair of disjoint circles. The involution τ acts on the torus by switching the two circles. Note that the involution $\tau \circ J$ fixes both circles pointwise.

8. Area and systole of a Klein surface

So far we have mostly dealt with conformal information independent of the metric. In this section we begin to consider metric-dependent invariants.

Lemma 8.1. *Given a J-invariant metric on the Klein surface $3\mathbb{RP}^2$, we have the following relations among the areas of the surfaces appearing in (6.3):*

$$\operatorname{area}(3\mathbb{RP}^2) = \operatorname{area}(S^2) = \operatorname{area}(K),$$

as well as

$$\operatorname{area}(\Sigma_2) = \operatorname{area}(\mathbb{T}_{a,b}) = 2\operatorname{area}(3\mathbb{RP}^2).$$

The systole, denoted "sys", of a space is the least length of a loop which cannot be contracted to a point in the space. Given a metric on a Klein surface

$$n\mathbb{RP}^2 = \Sigma_g/\tau,$$

we consider the natural pullback metric on its orientable double cover, denoted Σ_g. This metric is invariant under the involution τ. We define the least displacement invariant "disp" by setting

$$\operatorname{disp}(\tau) = \min \left\{ \operatorname{dist}(x, \tau(x)) \mid x \in \Sigma_g \right\}. \tag{8.1}$$

The systole of $\operatorname{sys}(n\mathbb{RP}^2)$ can be expressed as the least of the following two quantities:

$$\operatorname{sys}(n\mathbb{RP}^2) = \min \left\{ \operatorname{sys}(\Sigma_g), \operatorname{disp}(\tau) \right\},$$

where disp is the least displacement defined in (8.1).

9. Systolic estimates

Given a Riemannian metric on a Klein surface $n\mathbb{RP}^2$, we are interested in obtaining upper bounds for its systolic ratio

$$\frac{\operatorname{sys}^2}{\operatorname{area}}, \tag{9.1}$$

where "sys" is its systole. Recall that the following four properties of a closed surface Σ are equivalent:

(1) Σ is aspherical;
(2) the fundamental group of Σ is infinite;
(3) the Euler characteristic of Σ is non-positive;
(4) Σ is not homeomorphic to either S^2 or $\mathbb{RP}^2$.

The following conjecture has been discussed in the systolic literature, see [19].

Conjecture 9.1. *Every aspherical surface satisfies Loewner's bound*

$$\frac{\operatorname{sys}^2}{\operatorname{area}} \leq \frac{2}{\sqrt{3}}. \tag{9.2}$$

The conjecture was proved for the torus by C. Loewner in '49 and for the Klein bottle by C. Bavard [3]. M. Gromov [12] proved an asymptotic estimate which implies that every orientable surface of genus greater than 50 satisfies Loewner's bound. This was extended to orientable surfaces of genus at least 20 by M. Katz and S. Sabourau [21], and for the genus 2 surface in [22].

If the orientable double cover of the Klein surface is hyperelliptic, then its metric can be averaged by the hyperelliptic involution J so

as to improve (i.e. increase) its systolic ratio (9.1). This point was discussed in detail in [2]. Similarly, the ratio

$$\frac{\mathrm{disp}(\tau)^2}{\mathrm{area}}$$

defined in (8.1) is increased by averaging. Thus we may assume without loss of generality that the metric on the orientable double cover is already J-invariant.

Consider a systolic loop on $n\mathbb{RP}^2$, namely a loop of least length which cannot be contracted to a point in $n\mathbb{RP}^2$. The loop is either a 1-sided loop γ, or a 2-sided loop δ.

10. Outline of the argument

Assume that we are in the situation of a 2-sided loop $\delta \subset K$. Since δ lifts to a closed loop $\tilde{\delta}$ on the orientable double cover Σ_g, we have

$$\mathrm{sys}(n\mathbb{RP}^2) = \mathrm{sys}(\Sigma_g).$$

In the special situation $n = 3$ and $g = 2$, we proceed as follows. Consider the real model (6.1) of Σ_g, and its three companion tori of type (6.2).

For each companion torus, we pass to the quotient Klein bottle, and find a systolic loop satisfying Bavard's inequality (which is stronger than Loewner's bound). We thus obtain three loops $\delta_{a,b}, \delta_{b,c}, \delta_{a,c}$, and by our assumption the loops lift to loops $\tilde{\delta}_{a,b}, \tilde{\delta}_{b,c}, \tilde{\delta}_{a,c}$ on the torus. Each loop $\tilde{\delta}$ projected to the sphere S^2 defines a partition of the set of six branch points.

We consider the corresponding partitions of the set of six branch points on S^2. If the three partitions are not identical, then the corresponding loops $\tilde{\delta}$ can be rearranged by cutting and pasting (see next section), to give a loop on the Klein surface which satisfies Bavard's inequality.

In the remaining case, all three loops define identical partitions of the six branch points of Σ_g. This case will be handled in Section 15.

11. Cut and paste technique

The argument in [22] can be summarized as follows.

Lemma 11.1. *Given a genus 2 surface of unit area, one can find a pair of non-homotopic loops ℓ_1 and ℓ_2, of combined length*

$$|\ell_1| + |\ell_2| \leq 2C_{\mathrm{Loewner}},$$

where

$$C_{\text{Loewner}} = \sqrt{\frac{2}{\sqrt{3}}}$$

(see (9.2) *above). In particular, the shorter of the two is a Loewner loop on the surface.*

The proof can be summarized as follows. We exploit hyperellipticity to construct a pair of companion tori. We then apply Loewner's torus inequality to find Loewner loops on each of the two tori. Finally, we apply a cut and paste technique to rearrange segments of the two loops into a pair of loops that lift to the genus 2 surface.

Consider a systolic loop $\gamma \subset K$ of a Klein bottle K. A systolic loop is necessarily simple. It satisfies Bavard's inequality [3]

$$|\gamma| \leq C_{\text{Bavard}} \sqrt{\text{area}(K)},$$

where $C_{\text{Bavard}} = \sqrt{\frac{\pi}{\sqrt{8}}}$. Note that $C_{\text{Bavard}} < C_{\text{Loewner}}$ (Bavard's inequality gives a stronger bound than Loewner's).

Definition 11.2. A loop satisfying Bavard's inequality will be called a Bavardian loop.

12. Proof: 1-sided systolic loop

We now describe a procedure for constructing short loops on a Klein surface. Given a genus 2 surface (6.1), consider a companion Klein bottle

$$K_{a,b} = \mathbb{T}_{a,b}/\tau.$$

Consider a systolic loop on $K_{a,b}$. There are two cases to consider, according to the sidedness of the systolic loop. In this section, we consider the case when the systolic loop γ is one-sided, which turns out to be the easier of the two. Then γ lifts to a path connecting a pair of opposite points of the torus. Let $\tilde{\gamma}$ be its connected double cover on the torus $\mathbb{T}_{a,b}$.

Lemma 12.1. *The loop $\tilde{\gamma}$ contains a real point p such that the points p and $\tau(p)$ decompose $\tilde{\gamma}$ into a pair of paths*

$$\tilde{\gamma} = \gamma_+ \cup \gamma_-$$

such that each of γ_+, γ_- projects to a loop on S^2.

Proof. Note that $\tilde{\gamma}$ is invariant under the action of τ on the torus. The loop $\tilde{\gamma}$ projects to a loop

$$\tilde{\gamma}_0 \subset S^2.$$

The connected loop $\tilde{\gamma}_0$ is invariant under complex conjugation. Therefore it must meet the equator (see Section 7). Here the equator is the closure of $\mathbb{R}$ in $\hat{\mathbb{C}}$. A point $p_0 \in \hat{\mathbb{R}}$ where $\tilde{\gamma}_0$ meets the equator corresponds to a real point $p \in \tilde{\gamma}$ such that p and $\tau(p)$ have the same image in the sphere, namely p_0. Thus $\tilde{\gamma}_0$ necessarily has a self-intersection, unlike γ itself.

It may be helpful to think of $\tilde{\gamma}_0$ as a figure-eight loop, with its midpoint on the equator. In fact, the original loop γ on the bottle lifts to a path γ_+ joining p and $\tau(p)$, which then projects to half the loop $\tilde{\gamma}_0$, forming one of the hoops of the figure-eight. If we let $\gamma_- = \tau(\gamma_+)$, we can write

$$\tilde{\gamma} = \gamma_+ \cup \gamma_-.$$

Thus the loop $\tilde{\gamma}_0 \subset S^2$ is the union of two loops

$$\tilde{\gamma}_0 = P_x(\gamma_+) \cup P_x(\gamma_-),$$

where $P_x : \Sigma_2 \to S^2$ is defined (on the affine part) by the projection to the x-coordinate, see (2.2). $\square$

Next, we would like to transplant γ_+ to the Klein surface $3\mathbb{RP}^2$. The loop $P_x(\gamma_+) \subset S^2$ lifts to a path

$$\tilde{\tilde{\gamma}} \subset \Sigma_2$$

which may or may not close up, depending on the position of the third pair $(c, \bar{c})$ of branch points of $\Sigma_2 \to S^2$. If $\tilde{\tilde{\gamma}}$ is already a loop, then it projects to a Bavardian loop on the Klein surface $3\mathbb{RP}^2 = \Sigma/\tau$. In the remaining case, the path $\tilde{\tilde{\gamma}}$ connects a pair of opposite points $\tilde{p}, \tau(\tilde{p})$ on the surface Σ_2, where

$$P_x(\tilde{p}) = P_x(\tau(\tilde{p})) = p_0.$$

Therefore $\tilde{\tilde{\gamma}}$ projects to a Bavardian loop in this case, as well.

13. Proof continued: 2-sided systolic loop

It remains to consider the case when each systolic loop δ of each of the three companion Klein bottles

$$K_{a,b}, K_{a,c}, K_{b,c}$$

of the genus 2 surface (6.1) is 2-sided. Thus each of these Bavardian loops $\delta_{a,b}, \delta_{a,c}, \delta_{b,c}$ lifts to a closed loop on the corresponding orientable double cover. The resulting loop on the corresponding torus will be denoted $\tilde{\delta}$. The projection

$$P_x(\tilde{\delta}) \subset S^2$$

will be denoted $\delta_0 = P_x(\tilde{\delta})$.

Lemma 13.1. *If δ_0 meets the equator, then $3\mathbb{RP}^2$ contains a Bavardian loop.*

Proof. Let
$$p_0 \in \delta_0 \cap \hat{\mathbb{R}}.$$
Let $p, \tau(p) \in \Sigma_2$ be the points above it in Σ_2. We lift the path δ_0 to a path $\delta_+ \subset \Sigma_2$ starting at p. If δ_+ closes up, its projection to $3\mathbb{RP}^2$ is the desired Bavardian loop. Otherwise, the path δ_+ connects p to $\tau(p)$. In this case as well, δ_+ projects to a Bavardian loop on $3\mathbb{RP}^2 = \Sigma_2/\tau$. □

Thus we may assume that the δ_0 does not meet the equator of S^2.

Lemma 13.2. *Let δ be a systolic loop on a companion torus $\mathbb{T}_{a,b}$, and assume δ_0 does not meet the equator. Then δ_0 is a simple loop.*

Proof. By hypothesis, the loop δ_0 lies in a hemisphere. The typical case to keep in mind of a non-simple loop is a figure-eight. Without loss of generality, we may assume that δ_0 has odd winding number with respect to the ramification points
$$a, b \in S^2.$$
We will think of the curve δ_0 as defining a connected graph $\Delta \subset \mathbb{R}^2$ in a plane. The vertices of the graph are the self-intersection points of δ_0. Each vertex necessarily has valence 4. By adding the bounded "faces", we obtain a "fat" graph, denoted Δ_{fat} (the typical example is the interior of the figure-eight). More precisely, the complement $\mathbb{R}^2 \setminus \Delta$ has a unique *unbounded* connected component, denoted $E \subset \mathbb{R}^2 \setminus \Delta$. Its complement in the plane, denoted
$$\Delta_{\text{fat}} = \mathbb{R}^2 \setminus E,$$
contains the graph Δ as well as its bounded "faces". The important point is that both branch points a, b of the double cover $P_x : \mathbb{T}^2_{a,b} \to S^2$ must lie inside the connected region Δ_{fat}:
$$a, b \in \Delta_{\text{fat}}.$$
The boundary of Δ_{fat} can be parametrized by a closed curve ℓ, thought of the boundary of the outside component $E \subset \mathbb{R}^2$ so as to define an orientation on ℓ (in the case of the figure-eight loop, this results in reversing the orientation on one of the hoops of the figure-eight). Note that $\ell \subset \Delta$ is a subgraph. Since both branch points lie inside, the loop ℓ lifts to a loop on the torus which cannot be contracted to a point. If δ_0 is not simple, then ℓ must contain "corners" (as in the case of a figure eight-shaped δ_0) and can therefore be shortened, contradicting the hypothesis that δ is a systolic loop. □

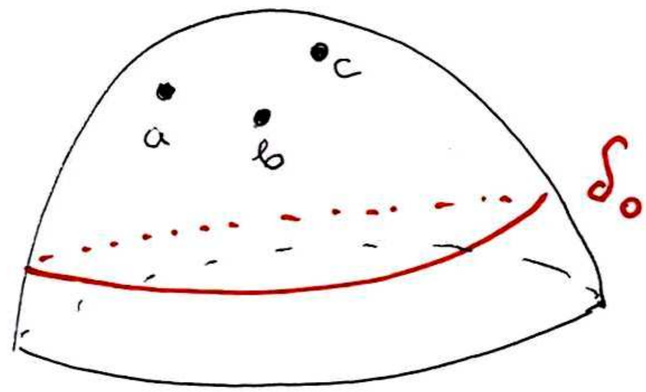

Figure 14.1. Klein surface as a hemispherical double cover

We may thus assume that the simple loop $\delta_0 \subset S^2$ separates the six points $a, \bar{a}, b, \bar{b}, c, \bar{c}$ into two triplets (a, b, c) and $(\bar{a}, \bar{b}, \bar{c})$. Hence its connected double cover in Σ_2 is isotopic to the equatorial circle of Σ_2. The latter is a double cover of the equator of S^2 (see Section 7).

14. Hyperellipticity for Klein surfaces

It may be helpful to think of the surface $3\mathbb{RP}^2$ as a double cover of the northern hemisphere of S^2, with the equator included. The cover is branched along the equator as well as at 3 additional branch points, namely the points a, b, c of the (standard) hyperelliptic cover $\Sigma_2 \to S^2$. Note that the three remaining branch points $\bar{a}, \bar{b}, \bar{c}$ are mapped to a, b, c by the involution τ.

A horizontal circle on S^2 with small positive latitude (south of all three branch points) is double-covered by a circle on $3\mathbb{RP}^2$ which can be thought of as the boundary of a Mobius strip, the central circle of which is the equator. The simple loop double covering δ_0 separates the surface into two surfaces with boundary: a torus with circular boundary, and a Mobius strip. The results of the previous section are summarized in the diagram of Figure 14.1.

Proposition 14.1. *The Klein surface* $3\mathbb{RP}^2$ *either contains a Bavardian loop, or admits a simple loop which separates it into a torus with*

a disk removed $\Sigma_{1,1}$ *and a Mobius strip* Mob*:*

$$3\mathbb{RP}^2 = \Sigma_{1,1} \cup_{S^1} \mathrm{Mob}$$

corresponding to a topological decomposition

$$3\mathbb{RP}^2 = \mathbb{T}^2 \# \mathbb{RP}^2,$$

such that moreover

- *the torus with a disk removed* $\Sigma_{1,1}$ *contains the three isolated branch points;*
- *the Mobius strip contains the equatorial circle;*
- *the separating loop is* J*-invariant;*
- *the separating loop is of length at most* $2C_{\mathrm{Bavard}}$*;*
- *the separating loop double-covers a loop*

  $$\delta_0 \subset S^2$$

  which lifts to a systolic loop on a companion torus.

15. Improving Gromov's 3/4 bound

Gromov's general 3/4 bound for aspherical surfaces,

$$\mathrm{area} \geq \frac{3}{4}\,\mathrm{sys}^2,$$

appeared in [12, Corollary 5.2.B]. We would like to improve the bound in our case of $3\mathbb{RP}^2$. The bound can be written as

$$\frac{\mathrm{sys}^2}{\mathrm{area}} \leq 1.3333\ldots$$

Theorem 15.1. *The bound*

$$\frac{\mathrm{sys}^2}{\mathrm{area}} \leq 1.333 \tag{15.1}$$

is satisfied by every metric on the Klein surface $3\mathbb{RP}^2$*.*

Note that the highest systole of a hyperbolic metric on $3\mathbb{RP}^2$ was identified by H. Parlier [26] to be $\operatorname{arccosh} \frac{5+\sqrt{17}}{2} = 2.19\ldots$ resulting in a systolic ratio of $.76\ldots$. The orientable double cover of this hyperbolic Klein surface is the Parlier-Silhol curve of genus 2 [26, 29].

Proof. A partition of $3\mathbb{RP}^2$ into a torus with a disk removed and a Mobius strip was constructed in Proposition 14.1. We normalize $3\mathbb{RP}^2$ to unit area. We will find a short loop on the torus with a disk removed if its area is at most .676, and on the Mobius strip if its area is at most .324.

Let $\alpha = 1.333$. To prove Theorem 15.1, we need to locate an essential loop of length at most

$$\beta = \sqrt{\alpha} \approx 1.15456.$$

If there is no such loop, the distance from each branch point to the equator must be at least $\frac{1}{2}\beta$. We apply the coarea formula to the distance function from the equator. Since the lift of δ_0 to the torus $\mathbb{T}_{a,b}$ is a systolic loop by Proposition 14.1, we obtain

$$2|\delta_0|\tfrac{1}{2}\beta \leq 1,$$

as the torus is normalized to unit area. Hence $|\delta_0| \leq \beta^{-1}$. The argument with attaching a hemisphere that we will present in Lemma 15.3 produces an estimate that becomes far more powerful as $|\delta_0|$ decreases. Therefore there is no loss of generality in assuming that equality takes place:

$$|\delta_0| = \beta^{-1} \approx .86613.$$

The theorem now results from the two lemmas below. □

Lemma 15.2. *If the Mobius strip has area at most* $|\mathrm{Mob}| \leq .324$ *then it contains an essential loop of square-length less than* $\alpha = 1.333$.

Proof. Let h be the least distance from a point of δ_0 to the equator. By the coarea formula applied to the distance function from the equator, we obtain

$$|\mathrm{Mob}| \geq 2h|\delta_0|. \tag{15.2}$$

Hence

$$h \leq \frac{|\mathrm{Mob}|}{2|\delta_0|} \approx .18704.$$

Connecting δ_0 to a nearest point of the equator by a pair of paths of length h, we obtain an essential loop on $3\mathbb{RP}^2$ of length at most

$$|\delta_0| + 2h = \beta^{-1} + 2\frac{|\mathrm{Mob}|}{2|\delta_0|} = \beta^{-1} + \beta|\mathrm{Mob}| \approx 1.26,$$

falling *short* (or, rather, *long*) of the required upper bound of $\beta \approx 1.15456$. We will improve the estimate (15.2) as follows. If

$$h \leq \frac{1}{2}(\beta - |\delta_0|) \approx \frac{1}{2}(1.15456 - .86613) \approx .14422,$$

then we obtain a short loop and prove the theorem. Thus we may assume that $h \geq .14422$.

A level curve at distance x from the equator must have length at least $\beta - 2x$ to avoid the creation of a short loop. Hence $|\mathrm{Mob}|$ is

bounded below by *twice* the area of a trapeze of altitude h, larger base β, and smaller base δ_0:

$$|\mathrm{Mob}| \geq h(\beta + \delta_0) = .14422\,(1.15456 + .86613) \approx .29,$$

which falls short of the required estimate .324.

Blatter [6, 7] and Sakai [28] provide a lower bound equal to the half the area of a belt formed by an h-neighborhood of the equator of a suitable sphere of constant curvature. Here the equator has length 2β while the its antipodal quotient is a Mobius strip of systole β. The radius of such a sphere is $r = \frac{2\beta}{2\pi} = \frac{\beta}{\pi}$. Hence the subtending angle γ of the northern half of the belt satisfies

$$\gamma = \frac{h}{r} = \frac{h\pi}{\beta} \approx .39241.$$

(here we use the values $\beta \approx 1.15456$ and $|\delta_0| = \beta^{-1} \approx .86613$). The height function is the moment map (Archimedes's theorem), and hence the area of the belt is proportional to

$$\sin\gamma \approx .38242.$$

The area of the corresponding region on the unit sphere is $4\pi \sin\gamma$. Hence the area of the spherical belt is

$$4\pi r^2 \sin\gamma = \frac{4\pi\beta^2 \sin\gamma}{\pi^2},$$

which after quotienting by the antipodal map yields a lower bound

$$|\mathrm{Mob}| \geq \frac{2\beta^2 \sin\gamma}{\pi} = \frac{2\alpha \sin\gamma}{\pi} = \frac{2(1.333)\sin\gamma}{\pi} \approx .32453 > .324,$$

proving the lemma. □

Lemma 15.3. *If the torus with a disk removed has area at most* .676*, then it contains an essential loop of square-length less than* $\alpha = 1.333$.

Proof. The separating loop is a circle of radius

$$r = \frac{2|\delta_0|}{2\pi} = \frac{|\delta_0|}{\pi}.$$

The area of a hemisphere based on such a circle is

$$2\pi r^2 = \frac{2|\delta_0|^2}{\pi} = \frac{2}{\pi\alpha}.$$

Attaching the hemisphere to the torus with a disk removed produces a torus of total area at most

$$\frac{2}{\pi\alpha} + .676.$$

Applying Loewner's bound (9.2) to the resulting torus, we obtain a systolic loop of square-length at most

$$\frac{2}{\sqrt{3}}\left(\frac{2}{\pi\alpha}+.676\right)\approx 1.33204<1.333,$$

proving the lemma and the theorem. $\square$

Remark 15.4. Gromov points out at the bottom of page 49 in [12] that his 3/4 bound can be improved by 1%; perhaps this estimate is what he had in mind.

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